of conversion is mathematics. As you get deeper and deeper into (the approved) mathematical studies, you come to think that the non-sensible things they deal with are not only context-invariant. They are also more real than anything you encounter in the fluctuating perspectives of ordinary life in the sensible world (515de). Admittedly, for a Platonist the Forms are yet more real and still more fundamental to explaining the scheme of things than the objects of mathematics. But already with mathematics we can see that abstract reasoning, understood in Plato's way as reasoning about a realm of abstract, non-sensible things, is reasoning about things which are themselves more real and more fundamental to explaining everything else. Mathematics provides the lowest-level articulation of the world as it is objectively speaking.

6. Abstract objects

What are these abstract, non-sensible items that mathematics reasons about? The question may be asked, and answered, at two levels: internal and external. By 'internal' I mean internal to the practice of mathematics itself. When you study arithmetic or geometry, what conception do you need of the objects (numbers, figures, etc.) you are dealing with? The external question is metaphysical: Where do these objects belong in the final scheme of things? What is their exact ontological status? We shall see that the Republic leaves the external question tantalisingly open. But readers are expected to find the internal question easy to answer. The chief clue is that Glaucian is supposed to know already, from his previous familiarity with mathematics.

Consider this famous passage (emphases mine):

'You will understand better after this preamble (ποικίλων προερ-

31 I think you know that the practitioners of geometry and arithmetic and such subjects start by hypothesising the odd and the even and the various figures and three kinds of angle and other things of the same family (ἀθέλομα) as these in each discipline. They make hypotheses of them as if they knew them to be true. They do not expect to give an account of them to themselves or to others, but proceed as if they were clear to everyone. From these starting points they go through the subsequent steps by agreement (ἀθέλομα) until they reach the conclusion they were aiming for.'

'Certainly I know that much', he said.

'Then you also know that they make use of visible forms and argue about them, though they are not thinking about these forms, but

32 In the phrase ποικίλων υποθέσεων αύτή the accusative αύτή refers to the three kinds of angle, etc., but this does not mean that mathematicians hypothesise things as opposed to propositions: see the survey of ποικίλων in C. C. W. Taylor, 'Plato and the Mathematicians: An Examination of Mr Hare's Views', Philosophical Quarterly, 17 (1967), 193–203.

33 Shorey translates 'consistently' here, but at 533c 5 he renders ἀποκλίτος by 'assent' or 'admission' and writes a note on how 'Plato thinks of even geometrical reasoning as a Socratic dialogue'. Most translators accept the desirability of using the same expression in both passages, but they divide into those who think that the point at 533c is that consistency is not enough for knowledge (so, most influentially, Robinson, Plato's Earlier Dialectic [2nd edn, Oxford, 1953], pp. 148 and 150) and those, like myself, who think the point is that knowledge or understanding should not depend on an interlocutor's agreement; all relevant objections should have been rebutted. The issue is too large to discuss here (it would involve a full investigation of the tasks of dialectic), but nothing in the present essay will depend on my preferred solution. Notice that in Book IV the principle of opposites, key premise of the proof that the soul has three parts, is accepted as a hypothesis for the discussion to proceed without dealing with all the objections that clever people might make, subject to the agreement that, if it is ever challenged by a successful counter-example, the consequences drawn from it will be 'lost', i.e. they must be regarded as unproven (437a). The parallel with the hypotheses of mathematics is quite close. All the other seven occurrences of ἀποκλίτος in Plato require to be translated in terms of agreement: Laches 186b 4, Laws 797b 7, Menexenus 243c 4, 245a 7, Symposium 186b 5, 196a 6, Theaetetus 157e 5, Proclus, Commentary on Plato's Republic 1 291.20 Kroll, writes of the soul being forced to investigate what follows from hypotheses taken as agreed starting-points (οι ἀρχαὶ ἀποκλίτος).
about those they are like. Their arguments are pursued for the sake of the square itself (τοῦ τετραγώνου ἀντίστοις) and the diagonal itself (διαμέτρου ἀντίστοις), not the diagonal they draw, and so it is with everything. The things they mould and draw — things that have shadows and images of themselves in water — these they now use as images in their turn, in order to get sight of those forms themselves, which one can only see by thought. ‘What you say is true’, he said. (510ce)

There is a lot here that Glauccon knows and we do not.

The mathematics of Plato’s day is largely lost, superseded by Euclid (c. 300 BC) and other treatises from the second half of the fourth century onwards. (The Republic was written in the first half of the fourth century.) However, Euclid’s Elements incorporates much previous work, from two main sources: first, earlier Elements by Leon and Theudius, both fourth-century mathematicians who spent time in the Academy; and second, the works of Theaetetus and Eudoxus, two outstanding mathematicians with whom Plato had significant contact. If we could read the mathematics available at the time Plato wrote the Republic, a good deal of it would look like an early draft of Euclid’s Elements. This does not quite get us back to the time when Glauccon studied mathematics, but the first Elements is credited to Hippocrates of Chios (c. 470–400 BC). (The dramatic date of the Republic is in the second half of the fifth century, no earlier than 432.) In any case, where stereometry and astronomy are concerned, Plato is obviously thinking of contemporary developments, not harking back to the fifth century; the same may well be true of the other mathematical disciplines. All in all, Euclid is now our best guide for contextualising the passage quoted. With due caution, therefore, let me present some Euclidean starting-points which seem to illustrate what Socrates says about mathematical hypotheses.

First, some of the geometrical definitions at the start of Elements I:

8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie on a straight line.
9. And when the lines containing the angle are straight, the angle is called rectilinear.
10. When a straight line set up on a straight line makes the adjacent angles equal, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
11. An obtuse angle is an angle greater than a right angle.
12. An acute angle is an angle less than a right angle.
13. A boundary is that which is an extremity of anything.
14. A figure is that which is contained by any boundary or boundaries.
15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point lying within the figure are equal to one another. 36

And so on for semicircle and the varieties of rectilinear figure (Elements I Defs 18–22). No elucidation, no account given of what these definitions mean or why they are true. The learner is expected to accept that these are the three kinds of angle and the various figures.

The presentation becomes still more abrupt if we subtract the neatly numbered tabulation of modern editions and translations. In the original, the arithmetical definitions that open Book VII would have looked more like this (without the bold type, spacing between words, and punctuation, which I keep as an aid to modern readers):

An unit is that in accordance with which (καθ’ ἑπτά) each of the things that exist is called one, and a number is a multitude composed of units. A number is a part of a number, the less of the greater, when it measures the greater, and parts when it does not measure it, and

34 The evidence for earlier Elements and their authors is Proclus, Commentary on the First Book of Euclid’s Elements, 66.20–68.10 Friedlein, relying (it is commonly agreed) on a history of mathematics by Aristotle’s pupil, Eudemus of Rhodes (second half of the fourth century). Plato died in 347 BC, so the time-gap is relatively small.

35 Lassere, De Léodamas de Thasos à Philippe d’Opunte, pp. 191–214 (Greek text), pp. 397–423 (translation), gives an impressive array of Euclidean starting-points already familiar to Plato and the Academy.


37 Here I follow Paul Pritchard, Plato’s Philosophy of Mathematics (Sankt Augustin, 1995), pp. 13–14, in rejecting Heath’s translation ‘that in virtue of which’, on the grounds that this suggests the unit is what makes something one, the cause of its unity. Aristotle in Metaphysics X inquires into what makes each of the things that exist one. Euclid merely presupposes they are each one.
the greater number is a multiple of the less when it is measured by the less. An even number is that which is divisible into two equal parts, and an odd number is that which is not divisible into two equal parts, or that which differs by an unit from an even number. An even-times even number is that which is measured by an even number according to an even number. 38

And so on for even-times odd number, odd-times odd number, prime number, numbers prime to one another, composite number and numbers composite to one another, etc., and finally perfect number (Elements VII Defs 9-22). Once again, Socrates' description is vindicated to a T. We may fairly hope that Euclid can also tell us something about what Glaucun knows about the mathematicians' use of visible forms.

In one respect, however, Euclid is likely to be misleading. The Elements is a book, and a long one at that. Diagrams can be included in a book, but not the moulded figures Socrates also mentions. 39 We will shortly hear of mathematical 'experts' laughing away an objection. That implies an oral presentation, which would be less formal than Euclid and would not include more initial hypotheses than were needed for the occasion. Much may be presupposed without explicit statement.

We should not exaggerate the difference this makes. Greek school-teaching was not child-oriented or kind. It included lots of dictation and rote-learning. 40 When Plato in the Republic has Socrates urge that play, not force, is the way to bring children into mathematics (536d—537a), he goes knowingly against the grain of the culture; in the Laws (819ac) the idea is presented as an import from Egypt. Equally innovating is the famous remark that sums up the message of the Cave. Education is not, as some people say, a

manner of putting knowledge into souls that lack it, like putting sight into blind eyes. The soul already possesses the 'instrument with which each person learns'. What is needed is to turn it around, as if it were an eye enfeebled by darkness, so that it can see invariant being instead of perspectival becoming (518bd). Part of the point of the mathematical scene in Plato's Meno is to contrast ordinary didactic instruction with the way Socrates gets the slave to see how to double the given square 'without teaching him', simply by his usual method of question and answer. And even Socrates starts out by asking whether the slave knows what a square is, namely, a figure like the one drawn which has all four sides equal (Meno 82bc).

I conclude that the oral teaching Glaucon is familiar with would reflect the formality of Euclid's procedure more closely than the education we are used to. In any case, the future rulers will not go on to their five years' dialectic until they have achieved a synoptic view of all the mathematical disciplines (Enigma A), and dialectic will centre on explaining the hypotheses of mathematics in a way that mathematics does not, and cannot, do (510b, 511b, 533c). For this purpose, not only the hypotheses of arithmetic and geometry, but also those of astronomy and harmonics, will need explicit formulation—all of them. In the long run, there will be no significant difference between oral and written mathematics.

It is the hypotheses that make it possible to use 'visible forms' (diagrams) to think about abstract, non-sensible objects. Socrates says that mathematicians argue about visible forms in order to reach results about something else. Without a more or less explicit idea of what that something else is, the procedure would be aimless. The visible forms mentioned are square and diagonal. Ancient readers would probably think at once of a geometer demonstrating the well-known proposition that the diagonal of a square is incommensurable with its side—no unit, however small, will measure both without remainder. 41 This example, a favourite

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38 Heath's translation still, but with 'and' inserted to mark each occurrence of the connective καί and the full stops indicating asymdecton in the sequel. In Book VII none of the MSS number the definitions; in Book I must do not. (I owe thanks to Reviel Netz for calling my attention to this fact, which can be verified by looking at the apparatus criticus of Heiberg's edition of the Elements [Leipzig: Teubner, 1883-8]).

39 Natural as it is to suppose the reference is to three-dimensional figures used in solid geometry, Timaeus 50ab speaks of moulding a piece of soft gold into a triangle and other (plane) figures.

40 Mura, Histoire, Part II, chaps 6-8.

41 An alternative, proposed by R. M. Hare, 'Plato and the Mathematicians'. in Renford Bambrough (ed.), New Essays on Plato and Aristotle (London, 1965), p. 25, is the square and diagonal drawn by Socrates in the Meno (82b-85a) to help the slave discover how to double the given square. But incommensurability lurks
with Aristotle too,\textsuperscript{42} makes good sense of Socrates’ observations, because the proposition is simply not true of the diagonal and side drawn in the diagram for the proof; to borrow a phrase from Ian Mueller, it is a proposition that ‘is always disconfirmed by careful measurement’.\textsuperscript{43} The geometer is well aware of that. He is using the diagram to prove something that holds for the square as defined in his initial hypotheses: ‘Of quadrilateral figures, a square is that which is both equilateral and right-angled’ (Elements I Def. 22). It has all four sides and all four angles exactly equal. That is what Socrates calls ‘the square itself’, the square represented (more or less accurately) by the diagram. He is, moreover, that it can only be seen in thought. The diagram representing this square is drawn ‘for the sake of’, as an aid to reasoning about, a square that the eyes do not see.

So far Socrates has said nothing that should surprise, nothing metaphysical, nothing with which Aristotle would disagree. His remarks articulate a conception of geometrical practice that any student of the subject must internalise. To an educated person like

Glaucón, it is familiar stuff.\textsuperscript{44} What is more, it is a conception of geometrical practice which supports Alcinous’ claim that the precision of mathematics is the essential epistemic route to a new realm of objects. Without a definition of square we would never be able to demonstrate a property such as incommensurability, which cannot be detected by the senses.

Visible forms were also used to diagram numbers. Here is the first proposition of Euclid, Elements VII:

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

For, the less of two unequal numbers $AB$, $CD$ being continually subtracted from the greater, let the number which is left never measure the one before it until an unit is left;

I say that $AB$, $CD$ are prime to one another, that is, that an unit alone measures $AB$, $CD$.

For, if $AB$, $CD$ are not prime to one another, some number will measure them.

Let a number measure them, and let it be $E$; let $CD$, measuring $BF$, leave $FA$ less than itself;

let $AF$, measuring $DG$, leave $GC$ less than itself,

and let $GC$, measuring $FH$, leave an unit $HA$.

Since, then, $E$ measures $CD$, and $CD$ measures $BF$, therefore $E$ also measures $BF$.

But it also measures the whole $BA$;

therefore it will also measure the remainder $AF$.

But $AF$ measures $DG$;

therefore $E$ also measures $DG$.

But it also measures the whole $DC$;

therefore it will also measure the remainder $CG$.

But $CG$ measures $FH$;

therefore $E$ also measures $FH$.

But it also measures the whole $FA$;

therefore it will also measure the remainder, the unit $AH$, though $E$ is a number: which is impossible.

Therefore no number will measure the numbers $AB$, $CD$;

therefore $AB$, $CD$ are prime to one another. Q.E.D.

\textsuperscript{42} At Prior Analytics I 23, 41a 26–7, Aristotle outlines a redactio proof which supposes that side and diagonal are commensurable and then shows how, in consequence, the same number will be both odd and even, which is impossible. Briefer allusions to the theorem at De Anima III 6, 430a 31 and other places listed in Bonitz, Index Aristotelicus (Berlin, 1870), 185a 7–16, with the comment ‘seipissine pro exemplo affertur’. The redactio proof is usually taken to be the one we read at Elements X, Appendix 27.


\textsuperscript{44} ‘That is why it helps him understand what Socrates was getting at in his first, densely compressed account of the upper two parts of the Divided Line (510b 4–9), to which Glaucón reasonably responded, ‘I don’t understand quite what you mean.’
laughable way. Of course, we could take as unit a smaller line—say, a fourth part of \( AH \). But now \( AH \) is four units instead of one ('If you cut it up, they multiply it'). The theorem is not falsified, merely inapplicable.

When Socrates speaks of 'the one itself' (cf. also 524e 6), he refers to something there are many of ('each of them equal to every other'), something that can be multiplied to compose a number. His 'one' is just like Euclid's 'unit', not a number but a component of number. Recall the first two definitions of Elements VII: the number is a multitude composed of units, where a unit (μικρόν) is 'that in accordance with which each of the things that exist is called one'. I understand this as follows.

Take anything that exists and think away all its features save that it is one thing. That 'abstracted' one thing is a Euclidean unit. Combine (in thought, of course — how else?) three such units, all absolutely alike (for there is nothing left by which they could differ), and you have a number — a three. Ancient arithmetic knows no such thing as the number three, only many sets of three units — many abstract triplets. It follows that, for a Greek mathematician, numerical equality is equinumerosity, not identity: '3 + 3 = 6' does not mean that the number 6 is identical with the number which results from adding 3 to itself, but that a pair of triplets contains exactly as many units as a sextet. For a more general illustration, consider Elements IX 35, where Euclid writes,

A similar interpretation in Jowett & Campbell's commentary (Oxford, 1894), ad loc., except that they imagine a schoolmaster gently laughing at a pupil's 'natural mistake' where I imagine the learner more contentious and the laughter as derisive. The learner is certainly not thinking of fractions, since at this period mathematicians studied (what we treat as) fractions as ratios between positive integers. Even Greek traders used only 2/3 and unit fractions of the form 1/n.

46 Cf. Theon of Smyrna, The mathematics which is useful for reading Plato, 18.18–21: ‘When the unit is divided in the domain of visible things, it is certainly reduced as a body and divided into parts which are smaller than the body itself, but if it is increased in numbers, because many things take the place of one’ (tr. Van Der Waerden).

47 The same idea at Philebus 56cc: whereas in practical arithmetic people count unequal units (two armies, two cows, etc.), theoretical arithmetic requires that one post (θήσει) a unit (μικρόν) which is absolutely the same as every other of the myriad units.
Let there be as many numbers as we please in continued proportion, $A$, $BC$, $D$, $EF$, beginning from $A$ as least, and let there be subtracted from $BC$ and $EF$ the numbers $BG$, $FH$, each equal to $A$;

I say that, as $GC$ is to $A$, so is $EH$ to $A$, $BC$, $D$.

The plural I have italicised: $A$, $BG$, and $FH$ are three different numbers, each equal to each other and each diagrammed separately in figure 3. By contrast, Heath's algebraic paraphrase is

$$(a_{n+1} - a_1) : (a_1 + a_2 + \ldots + a_n) = (a_2 - a_1) : a_1,$$

are the repeated use of a single symbol $a$, presupposes in the derrn manner that equal numbers are identical — a nice illustration for the thesis that it was the incorporation of algebra into in-stream mathematics during the Renaissance that created the derrn concept of number.$^{48}$

![Figure 3](image)

The Euclidean conception of units and numbers makes good use of what Socrates and Glaucion say in the last passage quoted. It is obviously true that Euclid's numbers can only be thought and not be handled in any other way. For the units that compose them require a deliberate act of abstraction: in each case your thought must set aside or ignore the many parts/features of the line of a pebble, bead on an abacus, or any other sensible object that might be to hand) in order to consider it as just: one thing. Once again, this would be conception of unit and number that any student would internalise. Glaucion already knows how experts answer the laughable suggestion. He can supply for himself (and for us) the mathematicians' answer to the question what kind of numbers they are talking about. To educated readers of the Republic it should all be familiar stuff.

What is more, Euclid's way of doing arithmetic is guaranteed to be virtually useless to traders (and modern accountants). He talks only of numbers that satisfy some general condition, never of 7, 123, or 1076; he never does what schoolchildren today call 'sums' or 'exercises'. 'Two unequal numbers being set out': they could be any unequal numbers whatsoever. That quest for generality marks the mathematician's desire for context-invariance.

7. The metaphysics of mathematical objects

But what, you may ask, are these units, numbers, and figures? Do they really exist, or are they just convenient posits to help us reason about objects still more rarefied and abstract, such as the Forms? That question — the external question — was certainly debated in the Academy, as we can tell from the last two Books of Aristotle's Metaphysics. There we learn that Plato and his associates, Speusippus and Xenocrates, each had their own answer, while Aristotle disagreed with the lot. But the question is not discussed in the Republic. In the two passages quoted in the previous section, Socrates is reporting what practising mathematicians do and say, not offering his own philosophical account of the ontological status of mathematical objects. In the next passage he says that such an account would be too much for the project in hand. After setting out the famous proportion between the various cognitive states represented in the Divided Line, 'As being (οὐσία) is to becoming (γένεσις), so is understanding (νόησις) to opinion (δόξα), and as understanding (νόησις) is to opinion (δόξα), so knowledge (ἐπιστήμη) is to confidence (πίστις) and thought (δύναμις) to conjecture (ἐκδοσία)', he adds: 'Let us leave aside the proportion exhibited by the objects of these states when the opinable (δοξαστόν) and the intelligible (νοητόν) are each divided into two.
Let us leave this aside, Glaucon, lest it fill us up with many times more arguments/ratios than we have had already (534a).

To refuse to contemplate the result of dividing the objects on the intelligible section of the Line is to refuse to go into the distinction between the objects of mathematical thought (διάνοια) and Forms. Pythagoras’ theorem (Euclid, Elements I 47), ‘In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle’, refers to three squares each of which, unlike the squares in Figure 4, has all four sides and all four angles exactly equal, as laid down in Elements I Def. 22. A theorem about three squares different in area cannot be straightforwardly construed as dealing with the (necessarily unique) Platonic Form Square, any more than the three equal numbers of Figure 3 can be construed as the (necessarily unique) Form of some number. The Republic tells us that practising mathematicians talk about plural, idealised entities which are not Forms. To judge by Euclid, this is true — a plain fact, which readers should be familiar with. About Forms the mathematicians need neither know nor care. Plato may have thought that the mathematicians’ multiple non-sensible particular numbers and figures (the ‘intermediates’ as they have been called in the scholarly literature since Aristotle) could ultimately be derived from Forms, so that in the end mathematics would turn out to be an indirect way of talking about Forms. Perhaps mathematical entities are the ‘divine reflections’ outside the cave (532c 1), dependent on the ‘real things’ they image. But whatever Plato thought, or hoped to show, Greek mathematics is quite certainly not a direct way of talking about Forms. If Plato has Socrates decline further clarification of the matter, we may safely infer that he supposed his message about mathematics and the Good could be conveyed without settling the exact ontological status of mathematical entities.

8. Controversial interlude

In denying that Plato thinks mathematics is directly about Forms, I am taking a controversial line. I should say something to pacify scholars who suppose otherwise. Two sentences have been influential in encouraging the interpretation I reject:

(1) ‘Their arguments are pursued for the sake of the square itself (τοῦ τετραγώνου αὐτοῦ ἑνακα) and the diagonal itself (διαμέτρου αὐτῆς), not the diagonal they draw.’ (510de, p. 24 above)
(2) ‘If someone tries by argument to divide the one itself (αὐτὸ τὸ ἑν), they laugh at him and won’t allow it.’ (525de, p. 30 above; cf. also αὐτὸ τὸ ἑν ἀτ 524c 6)

The issue is whether that little word ‘itself’ signals reference to a Platonic Form, as in phrases like ‘justice itself’ (517e 1–2), ‘beautiful itself’ (507b 5), or ‘the equal itself’ (Phaedo 74a 11–12).

The word ‘itself’ is certainly not decisive on its own, otherwise a
Form of thirst would intrude into Book IV's analysis of the divided soul. When Socrates there speaks of 'thirst itself' (437e 4: αὐτῷ τὸ δὲ φρίτη), he means to pick out a type of appetite in the soul, not a Form; in context, the phrase is equivalent to his earlier locution 'thirst qua thirst' (437d 8: καθ' ἑαυτὸν δύσμα ἔστι). Even the intensified expression 'itself by itself' (αὐτῷ καθ' αὑτῷ), which often signals a Platonic Form (e.g. 476b 10–11, Phaedo 100b 6, Symposium 211b 1, Parmenides 130b 8, 133a 9, c 4), does not always do so. Otherwise, when Socrates in the Phaedo recommends using 'pure thought itself by itself' to try to hunt down each pure being itself by itself' (66a 1–3), he would be telling one Form to study another. In Plato 'itself' and 'itself by itself' standardly serve to remove some qualification or relation mentioned in the context. Their impact is negative. Only the larger context will determine what remains when the qualification or relation is thought away. When the Phaedo (74a) distinguishes 'the equal itself' from 'equal sticks and stones', what remains is indeed a Form. But when Adeimantus in the Republic (363a) complains that parents and educators of the young do not praise justice itself (αὐτῷ δικαιοσύνη), only the good reputation you get from it, 'justice itself' does not yet signify a transcendent Platonic Form.\(^{51}\) And when in the Theaetetus the well-known fallacious argument against the possibility of judging what is not is framed within a distinction between 'what is not itself by itself' and 'what is not about something that is' (188d, 189b), it is definitely not the Sophist's Form of Not-Being that remains; it is a blank nothing, which no one could judge.

Now in (1) 'the diagonal itself' is opposed to 'the diagonal they draw', in (2) 'the one itself' contrasts with a one composed of many parts. In both cases the larger context is mathematics, not metaphysics. It is to mathematics, then, that we should look to judge the effect of the word 'itself'. In (1) it tells us to ignore the wobbles in the drawing and the fact that the line has breadth, in (2) to abstract from the many parts of the item we take as unit.\(^{52}\) Any page of Euclid shows that that is how mathematicians proceed. What remains when they do so is not a Form, but an ideal exemplification of the relevant definition.

Another standard view I reject is that Socrates means to criticise the mathematicians for the procedures he describes.\(^{53}\) 'Plato's criticism of the mathematicians' is a staple of the scholarly literature. The most influential sentence here is

> (3) 'They make hypotheses of them as if they knew them to be true. They do not expect to give an account of them to themselves or to others, but proceed as if they were clear to everyone.' (510c, p. 23 above)

Now mathematical thought (διάνοια) is twice characterised as a state in which the soul is forced (ἀναγκάζεται) to make use of hypotheses (510b 5, 511a 4; cf. 511c 7). It would seem harsh to pillory the mathematicians for doing something they are forced to do.

Why are they forced to use hypotheses? Plato's answer, I suggest, is that hypotheses are intrinsic to the nature of mathematical thought. There is no other way of doing deductive mathematics than by deriving theorems and constructions from what is laid down at the beginning. The very idea of an Elements is to find the simplest and most primitive starting-points from which the rest can be derived; that is what the title Στοιχεῖα means.\(^{54}\) To demand

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\(^{51}\) Nor does it even at 472c, where justice itself, the virtue they have been trying to define, is contrasted with the perfectly just man of Glaucous's challenge in Book II (306e–306d); see Adam's commentary (Cambridge, 1902), ad loc. The Theory of Forms makes its first appearance in the Republic, complete with the Phaedo's technical terminology of participation, at 475e–476d. Socrates starts by saying it would not be easy to explain to someone other than Glaucous. That marks the context as more metaphysical than the earlier ones. In such a context, a phrase like 'the beautiful itself' does indicate a transcendent Platonic Form.

\(^{52}\) Compare 'five and seven themselves (αὐτῷ πέντε καὶ ἑπτά) vs 'seven men and five men' at Theaetetus 195c–196a. The latter are objects of perception, the former can only be grasped in thought, yet in the context they cannot be Forms.

\(^{53}\) A leading exponent of this view was Richard Robinson, Plato's Earlier Dialectic, pp. 146–56, according to whom Plato criticises the mathematicians for failing to treat their starting-points as hypotheses: they take them as evident and known when they should regard them as tentative hypotheses. Robinson's account is echoed in Anaxas, Introduction to Plato's Republic, pp. 277–9, and many others.

\(^{54}\) See Walter Burkert, Ἀριστοτελεῖα: Eine semasiologische Studie', Philologus, 303 (1959), 167–97, where the theory that Στοιχεῖα originally meant the letters of the alphabet is finally laid to rest. What Euclid was admired for was not original mathematical results, but his skill at systematising the results of creative mathematicians like Thales and Eudoxus: see the introductory scholia to Elements V and XIII (282.13–20 and 654.1–10 Heiberg-Menge).
at the mathematicians give an account of their initial hypotheses, themselves and others, would be to make them stop doing mathematic and do something else instead. The best and brightest the Guards will indeed do that later. They will stop treating mathematical hypotheses as starting-points (511b 5; ἀρχής) and try to account for them in terms of Forms (511bc, 533c). But this activity is dialectic, not mathematics reformed to meet a criticism. Socrates expressly says that only dialectic can do the job (533c), the soul engaged in mathematical thought cannot (511a 5–6); and laucouc knows very few professional mathematicians who are so skilled in dialectic (531dc). It is thus no criticism to say that mathematicians give no account of their hypotheses. It is simply to say that mathematics is what they are doing, not dialectic.55

Another influential passage is where Socrates mocks the language of geometry:

‘This at least’, I said, ‘will not be disputed by those who have even a slight acquaintance with geometry, that this science is in direct contradiction with the language its practitioners use in their arguments.’

‘How so?’ he said.

‘They talk in a way that is both quite ludicrous and unavoidable (μάλα γελοῖος περὶ τὰ ἄλλα γεγονός). They speak as if they were doing something and developing all their arguments for the sake of action. They use words like “to square”, “to apply”, “to add”, and so on, whereas in fact the entire study is pursued for the sake of knowledge.’

‘That is so’, he said.

‘Then must we not agree on a further point?’

‘What?’

‘That this knowledge at which the study of geometry aims is

knowledge of what always is,56 not of what at a particular time comes to be and perishes.’

‘That is readily admitted’, he said. ‘Geometry is knowledge of what always is.’ (527ab)

A good illustration for these remarks is the way Euclid sets about proving Pythagoras’ theorem (Elements I 47) with the aid of Figure 4 (square-bracketed references are to earlier results used on the way):

Let ABC be a right-angled triangle having the angle BAC right.
I say that the square on BC is equal to the squares on BA, AC.

For let there be described on BC the square BDEC, and on BA, AC the squares GB, HC; through A let AL be drawn parallel to either BD or CE, and let AD, FC be joined.

Then, since each of the angles BAC, BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC, AG not lying on the same side make the adjacent angles equal to two right angles;

therefore CA is in a straight line with AG. [I 14]

For the same reason
BA is also in a straight line with AH.

And, since the angle DBC is equal to the angle FBA: for each is right:

let the angle ABC be added to each:

therefore the whole angle DBA is equal to the whole angle FBC. [Common Notion 2]

And, since DB is equal to BC, and FB to BA,
the two sides AB, BD are equal to the two sides FB, BC respectively, and the angle ABD is equal to the angle FBC;

therefore the base AD is equal to the base FC,

and the triangle ABD is equal to the triangle FBC. [I 41]

Now the parallelogram BL is double of the triangle ABD,
for they have the same base BD and are in the same parallels BD, AL.

And the square GB is double of the triangle FBC,

56 Shorey and some other translators miss the point that this clause is governed by the preceding εὐεκα.
57 I doubt Plato means Glaucon to do more here than affirm the first of Socrates’ alternatives. Glaucon grasped at 511cd that mathematics without dialectic is not knowledge in the fullest sense, but Socrates has just spoken of geometry as a science (527a 2: ἐπιστήμη).
for they again have the same base $FB$ and are in the same parallels $FB$, $HC$. [I 41]

Therefore the parallelogram $BL$ is also equal to the square $GB$.

Similarly, if $AE$, $BK$ be joined,

the parallelogram $CL$ can also be proved equal to the square $HC$;

therefore the whole square $BDEC$ is equal to the two squares $GB$, $HC$. [Common Notion 2]

And the square $BDEC$ is described on $BC$,

and the squares $GB$, $HC$ on $BA$, $AC$.

Therefore the square on the side $BC$ is equal to the squares on the sides $BA$, $AC$.

Therefore in right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle. Q.E.D.

Notice that the diagram is constructed step by step before the argument begins at ‘Then, since . . . ’; only $AE$ and $BK$ are added later. Euclid starts by asking us to accept that $ABC$ is a right-angled triangle (as defined at Elements I Def. 21) and then asks us to agree to his describing squares on each of its three sides (an operation licensed by the immediately preceding Elements I 46). Finally, he asks to draw various lines (licensed by Elements I Postulate 1, ‘To draw a straight line from any point to any point’). These lines are not mentioned in the proposition, which asserts a relationship between the squares on the sides of a right-angled triangle. But they are crucial to the proof, for they create the triangles ($ABD$, $FBC$) and parallelograms ($BL$, $CL$) on which the argument will turn. Not to accept them would be to deny the reality of the continuum, which is a presupposition of every proof in the book. Without the activity of drawing them, the proof could not get started. Likewise, without the (non-physical) action of adding the angle $ABC$ to each of the angles $DBC$ and $FBA$, the proof could not be continued. Socrates is right to say that the verbs of action (‘let $AL$ be drawn’, ‘let $AD$, $FC$ be joined’, ‘let the angle $ABC$ be added’) are unavoidable. Banishing them would be the death of (Greek) geometry.

But he is having fun when he says they are ludicrously at odds with the aim of the subject, which is to gain knowledge of invariant being. The theorem proved is an eternal, context-invariant truth.

What takes place in time is only the process of coming to know it is true by drawing the lines and conducting the proof. And it is typical of human learning in general, not peculiar to geometry, that it takes time and effort. Even the arithmetical proof at Elements VII I (quoted above) involves the operation of continual subtraction. We should not mistake a joke for serious criticism.

Admittedly, while the hypotheses remain unaccounted for, mathematics does not rank as knowledge or understanding in the fullest sense (511cd, 533c). By providing such accounts, in the light of a first principle (the Good), dialectic will give the subject-matter of mathematics an intelligibility that mathematics on its own cannot achieve: mathematics studies things that are ‘intelligible with the aid of a (first) principle’ (511d 2: νοητὰ μετὰ ἀρχῆς). But all that follows from this is that the mathematicians would be open to criticism if they claimed to know that their hypotheses are true. In (3) Socrates does not suggest that they do claim this, only that they proceed as if they knew them to be true and as if they were clear to everyone. To judge by the quotations I gave earlier from the opening of Euclid’s Elements I and VII, Socrates has it exactly right. Euclid never claims to know, but proceeds as if he did. He does not claim that his definitions are clear to everyone, but he proceeds as if they were. That is how (Greek) mathematics is done. Criticism is beside the point. Still less should anyone call upon Euclid to reform his mathematics. What Socrates is asking Glaucon to do (and through him, readers of the Republic) is something quite different: to agree that his description of mathematical procedures is plain fact, familiar stuff, and to reflect on the epistemological peculiarity of mathematics as such.

Glaucon understands this pretty well:

‘I understand’, he said, ‘though not adequately, for it is no slight task you appear to have in mind. You mean to say that the region of intelligible being which is contemplated by dialectical knowledge is clearer than the part studied by the arts (so called) which use hypotheses as starting points. Mathematicians are forced to contemplate their objects by thought (διανοία) rather than perception,

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38 A point insisted upon by Speusippus: Proclus, Commentary on the First Book of Euclid’s Elements, 77.15–78.8 Friedlein; Aristotle, De Caelo I 10, 279b 32–280a 2.
but because they study them from hypotheses, without having gone back to a (first) principle,\(^5^9\) you do not think they have understanding (\(\nu\omicron\omicron\omicron\nu\)) of them, even though they are intelligible with the aid of a principle. And I think you call the cognitive state (\(\epsilon\xi\nu\)) of the geometers and other mathematicians thought (\(\delta\iota\alpha\omicron\omicron\omicron\omicron\alpha\nu\)), not understanding (\(\nu\omicron\omicron\omicron\nu\)), because you take it to be intermediate between opinion (\(\delta\omicron\epsilon\omicron\xi\eta\)) and understanding.'

'You have got the point', I said, 'quite adequately.' (511cd)

That is the main result of the Divided Line passage: the introduction of a new intermediate epistemic state, which turns out to have an intermediate degree of clarity when it is compared, on one side with the ordinary person’s opinion about sensibles, and on the other side with the dialectician’s understanding of Forms. Socrates can then correlate this intermediate degree of cognitive clarity with the intermediate degree of truth or reality which belongs to the non-sensible objects that mathematicians talk about (511de). In sum, mathematics is not criticised but placed. Its intermediate placing in the larger epistemological and ontological scheme of the Republic will enable it to play a pivotal, and highly positive, role in the education of future rulers.

9. Values in the Cave

This brings me back to mathematics as the lowest-level articulation of the world as it is objectively speaking. The next step is to bring value into the picture. For that we must return to the Cave.

The prisoners, remember, are immobilised by chains which stop them seeing anything but the shadows on the back of the cave. The shadows are cast by firelight playing on a series of objects and puppet-like figures (human and animal) which are carried, unseen by the prisoners, along the top of a low wall behind them. The story starts when one of these prisoners is untied and forced to turn around to answer questions about the objects on the wall. That turning around (\(\pi\epsilon\iota\alpha\gamma\omega\gamma\eta\)) — 515c 7, 518d 4, e 4, 521c 6), or conversion (\(\mu\epsilon\tau\alpha\sigma\rho\omicron\omicron\phi\nu\)) — 518d 5, 525a 1, 526e 3, 532b 7), is the first stage of a long arduous journey which takes the freed prisoner up out of the cave into the brightly lit world outside, where their eyes gradually adjust, first to seeing shadows, then reflections, then the actual people and things reflected, then stars and moon (by night, of course), and finally the sun itself (515c—516b).

The story continues with an account of what happens to people who return to the cave (516e—518b). It is during this second phase that Socrates tells us to apply the whole Cave image to the two preceding images, the Sun and Divided Line (517ac). Specifically, we should give the sun outside the cave the same role as it had earlier in the analogy of the Sun: in both Sun and Cave the sun represents the Form of the Good (508e 2–3, 517b 8–c 1: \(\eta\) \(\tau\omicron\omicron\) \(\delta\gamma\alpha\theta\omicron\omicron\omicron\omicron\ \iota\omicron\omicron\alpha\)). We have long been aware that the Form of the Good is the 'greatest study' (505a 2: \(\mu\epsilon\gamma\iota\omicron\omicron\omicron\omicron\ \mu\alpha\theta\omicron\omicron\omicron\omicron\omicron\)). Because knowledge of it is the only sure guide to living well and enjoying the benefits of justice (505ab, repeated here at 517c). That much is clear (at least in outline): the goal and climax of the education that Socrates and Glaucon are planning for the rulers of the ideal city is knowledge of the Good.

Less clear is how we should understand the objects seen by the freed prisoner on the way up past the low wall to the world outside. What do the puppets on the wall represent, or the reflections outside the cave? And what is the significance of the fact that both puppets and reflections are likenesses of the animals themselves and other originals in the upper world? Socrates instructs us to apply the prisoner's upward journey to the soul's ascent into the intelligible region of the Divided Line (517b 4–5). This at once suggests mathematics and dialectic, with their respective objects, as described at the end of Book VI. In that case, the animals and other originals will represent the Forms studied by dialectic, while both reflections and puppets will be mathematical objects (perhaps conceived at different levels of abstraction). But the surrounding narrative, about the journey back to the cave, would suggest a different solution.

For the examples mentioned in the story are values. When

\(^5^9\) Note the aorist \(\delta\nu\epsilon\lambda\delta\omicron\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\n
someone goes down into the cave again, to begin with, before their
eyes have adjusted to the semi-darkness, they will appear ridiculous
if they have to dispute in court or elsewhere about the shadows of
the just, or about the puppets those shadows derive from, in terms
intelligible to people who have not seen justice itself (517de). Later,
however, after getting used to the poor light, they will do much
better than the prisoners at knowing what the shadows are and
knowing what they are shadows of. They will do better at this
precisely because they have seen the truth about what is beautiful,
just, and good (520c). Thus at least some of the shadows, hence at
least some of the puppet-like figures carried along the wall,
represent values like justice. If so, the same must be true of the
corresponding reflections and their originals outside the cave. The
conversion and ascent is progress towards an understanding of true
values.

One important difference between the puppets inside the cave
and the reflections outside is that the puppets are manipulated by
people behind the wall. Some of these puppeteers speak, but their
voices echo off the back of the cave in such a way that the prisoners
suppose they come from the shadows in front of them. Other
puppeteers remain silent: the effects they produce are purely visual
(514b–515b). This distinction suggests to me that among
the puppeteers are the poets and painters who transmit the values of
the community. The idea will be that the prisoners’ experience of
those values is mediated by the culture they grow up in. To get
outside the cave is to transcend one’s culture and achieve a more
objective understanding of justice, beauty, and goodness.

But that does not settle the question: how, by what studies, this
progress is achieved. The story indicates that innate to every
prisoner is an instrument (ἀγαθόν) capable of understanding
true values, but that to activate this capacity the whole personality
must be turned around, away from the world of becoming, so as to
redirect the ‘eye of the soul’ towards the realm of true being (518b–
519b). Socrates then asks (521c), ‘What studies will have that
effect?’ The question is open. The answer, of course, is mathematics,
but Socrates has to argue it at length (522b–531d). In effect, he is
arguing that an education in advanced mathematics is progress
towards understanding true values. At the end, after five mathe-
matical disciplines have been selected for the curriculum, he sums
up: these are the studies that will effect the conversion and the
ascent to the objects on the wall and the journey up out of the cave
as far as the reflections outside (532bd). Only the last stage,
represented in the simile by looking at the people and other real
things outside, is reserved for dialectic. I conclude that mathematics
provides the lowest-level articulation of objective value.

It will not do to object that values need not enter the story until
the rulers-to-be reach dialectic. If there are puppets representing
justice, and mathematics is what takes the freed prisoners to the
objects on the wall, then mathematics already gives them a better
understanding of justice than they had before, even if they do not
realise this until they come back down again. In the poetic narrative
of the Cave, the first thing that happens after the prisoners are
released from their chains is that they are shown the puppets one by
one and forced to answer the question ‘What is it?’ (515d). In the
retrospective prose of the mathematical curriculum, the first ques-
tion they are forced to confront is ‘What kind of numbers are the
mathematicians talking about?’ (525d–526b, quoted on p. 30
above). It would surely take lots of mathematics and much philo-
sophising to convince one that pure numbers are the key to debates
about justice in court or assembly. That insight should be reserved
for prisoners who have made the ascent and then returned. It is
important here that even the higher level of dialectic turns out to
have a strongly mathematical content.

Dialectic is not deductive proof, but philosophical discussion
aimed at testing and securing definitions (533ab), and we have
already seen that what the future rulers are to discuss in this way is
the hypotheses they relied on when doing mathematics (511b, 533c).
It is these that will lead to the unhypothetical first principle of
everything, the Good (511b, 533bd). Just how they will lead up to knowledge of the Good is a difficult and debated question in the scholarly literature. For present purposes, it is enough that dialectic is described in terms that suggest what we might call a meta-mathematical inquiry. The education of the rulers is mathematical, in one sense or another, all the way to the top. The famous image of dialectic as the coping stone (θρεγκός) of the curriculum (534e) implies the completion of a single, unified building, not a transfer to different subjects in a different building.

Yet the education of the rulers is also, from beginning to end, about value. At the beginning, as we have seen, they meet puppets of the just (note the plural ἄγαλματα at 517d 9); at the end the Good, which Socrates describes both as the cause of all things right and beautiful, and as that which anyone who is going to act wisely either in private or in public life must know (517c). They return having seen ‘justice itself’ (517e). But when? Plato could easily have made Socrates say that dialectic involves both trying to account for mathematical hypotheses in terms of Forms and discussing the Form of Justice.62 Instead, he leaves us to infer that dialectical debate about the conceptual foundations of mathematics is itself, at a very abstract level, a debate about values like justice. I think the inference is correct. The mathematics and meta-mathematics prescribed for the future rulers is much more than instrumental training for the mind. They are somehow supposed to bring an enlargement of ethical understanding. My final question is, How could that be?

62 F. M. Cornford, ‘Mathematics and Dialectic in the Republic VI–VII’, Mind, 41 (1932), 37–52 and 173–90, cited from R. E. Allen, Studies in Plato’s Metaphysics (London & New York, 1965), chap. 5, 50ff., argued that Plato divides the description of dialectic into two parts, one about mathematical dialectic and mathematical Forms (533a–534b), the second about moral dialectic and moral Forms (534bd), each part having its own distinctive methodology. His argument has not won acceptance, and in any case Socrates implies that a dialectical account of the Good will be of the same type, subject to the same tests, as the dialectical account of anything else (534b 8: ὀσοντοσ).

10. Harmonics

The place to start looking for an answer, I suggest, is the discussion of harmonics. When Socrates insists that mathematical harmonics should ‘ascend to problems to consider which numbers are concordant, which are not, and why each are so’, Glaucon exclaims, ‘You are speaking of a task which is superhuman (δαιμονικόν πράγμα).’ Socrates corrects him: ‘Say rather, a task which is useful if directed towards investigating the beautiful and good, but useless if otherwise pursued’ (531c). Both Pythagorean harmonics and Plato’s are concerned with concord (συμφωνία). The difference is whether they seek concords in heard sounds or at a more abstract level. Socrates implies that moving to the more abstract level is a prerequisite for harmonics to help us understand values like beauty and goodness. At that level, the answer to the question ‘Why are these numbers, unlike others, concordant?’ cannot be that they determine intervals which sound good to the ear. So what kind of explanation can Plato have in mind? That was Enigma C.63

As before, the best guide is Euclid. In the preamble to his Sectio Canonis we find this:

Among notes we recognize some as concordant, others as discordant, the concordant making a single blend out of the two, while the discordant do not. In view of this it is reasonable (εἰκός) that the concordant notes, since they make a single blend of sound out of the two, are among those numbers which are spoken of under a single name in relation to each other, being either multiple or epimoric. (149.17–24 Jan)64

This preliminary remark relies on a feature of the vocabulary the Greeks used to speak of ratios. For the multiple ratios 2:1, 3:1, 4:1, etc., they had one-word expressions, ending in -πλασιος, just like our ‘double’, ‘triple’, ‘quadruple’, and so on. Unlike us, they also

63 P. 14 above.
64 Tr. Barker, GMW II, p. 193, but with Mueller’s rendering of εἰκός as ‘reasonable’ substituted for Barker’s ‘to be expected’ (Mueller, ‘Ascending to Problems’, p. 113). It is kinder to Euclid to have him talk of what ought to be, rather than of what can in fact be expected in advance. Kinder still, and linguistically permissible, would be ‘appropriate’.
had a series of word expressions for epimoric ratios, which are ratios of the form \( n+1:n \). Thus 3:2 (the ratio of the fifth) is \( \nu \mu \nu \delta \lambda \omega \nu \), meaning ‘half-and-whole’; 4:3 (the ratio of the fourth) is \( \epsilon \pi \tau \rho \tau \rho \tau \rho \), meaning ‘third-in-addition’; 5:4 is \( \epsilon \pi \tau \tau \tau \tau \tau \tau \) (‘fourth-in-addition’), and so on. Other ratios, by contrast, collectively called ‘epimeric’, had no such expression assigned to them in the language, but were specified long-windedly as, e.g., ‘seven to four’. Euclid’s idea, then, is that Greek gives apt recognition to the unity of sound in a concord by assigning a single expression to the corresponding mathematical ratio.  

Whatever we think of this linguistic observation, it is clearly not an explanation of which numbers are concordant and which are not, and there is no reason to think that Euclid meant it as an explanation. For lots of multiple and epimoric ratios produce discordant intervals. But in the *Sectio Canonis* he does assume, when the mathematics gets going after the preamble, that concordant ratios are all either multiple or epimoric. That assumption was devotedly maintained in the tradition of mathematical harmonics to which Euclid belongs, despite a notorious difficulty caused by the interval of octave plus fourth. This is concordant to the ear, but its ratio is 8:3, which is neither multiple nor epimoric. The choice before a theorist was either to modify their mathematics or to say, in Platonic style, ‘So much the worse for empirical perception’. Euclid deftly escapes the dilemma by not mentioning this interval anywhere. But he is useful for our purposes in two ways. First, the *Sectio Canonis* is an example of the sort of mathematics Plato will have had in mind when he called for an investigation, by means of problems, of which numbers are concordant and which are not. Second, the assumption that concordant ratios are all either multiple or epimoric may provide at least a glimpse of the sort of explanation he wanted of why certain numbers are intrinsically concordant.

But Euclid is around half a century later than Plato. 66 If we track back to the time when the *Republic* was written, it seems that Glaucous, knowledgeable though he is about music (398e; cf. 548de), is not familiar with any mathematical treatment of the subject. For when Socrates refers to Pythagorean harmonics as an approach to music which goes wrong in the same way as the calendric astronomy he castigated earlier, Glaucous does not recognise the allusion. He supposes that Socrates means an empirical, string-torturing approach which gives the ear primacy over reason, not a mathematics which seeks numbers in heard concords. Socrates has to explain that he means ‘the Pythagoreans’ he mentioned earlier (530e–531c), i.e. Archytas. I infer that readers of the *Republic* are not expected to be familiar with Archytas’ mathematical harmonics; it is recherché stuff.

But it was known to Euclid. Proposition 3 of the *Sectio Canonis*, ‘In the case of an epimoric interval, no mean number, neither one nor more than one, will fall within it proportionally’, was first proved by Archytas. 67 We can hope that Archytas may offer further help with Enigma C. Here, then, is some more Archytas:

There are three means in music. One is arithmetic, the second geometric, the third subcontrary, which they call ‘harmonic’. There is an arithmetic mean when there are three terms, proportional in that they exceed one another in the following way: the second exceeds the third by the same amount as that by which the first exceeds the second. In this proportion it turns out that the interval [i.e., the musical interval] between the greater terms is less, and that between the lesser terms is greater. There is a geometric mean when

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65 This is not the only place where Euclid shows an interest in names. At *Elements* VII 37 he proves the trivial-seeming proposition, ‘If a number be measured by any number, the number which is measured will have a part called by the same name (\( \nu \mu \nu \delta \lambda \omega \nu \)) as the measuring number’, and at VII 38 that ‘If a number have any part whatever, it will be measured by a number called by the same name as the part’. Note once again the non-modern idea that the part and the measuring number are distinct, not one and the same number.

66 Assuming the *Sectio Canonis* is by Euclid. But the attribution is debated; see André Barbera, *The Euclidean Division of the Canon: Greek and Latin Sources* (University of Nebraska Press, 1991), pp. 3–36. If the treatise is not by Euclid, its date becomes uncertain, but it is still the best means to contextualise Plato’s discussion of concord.

they are such that as the first is to the second, so is the second to the third. With these the interval made by the greater terms is equal to that made by the lesser. There is a subcontrary mean, which we call 'harmonic', when they are such that the part of the third by which the middle term exceeds the third is the same as the part of the first by which the first exceeds the second. In this proportion the interval between the greater terms is greater, and that between the lesser terms is less. (Archytas frag. 2 Diels-Kranz)\(^{68}\)

The musical significance of these means may be illustrated as follows.

(i) The three numbers 12, 9, 6 are in arithmetical proportion, and 9 is the arithmetical mean between 12 and 6, because 12 exceeds 9 by the same amount as 9 exceeds 6. The ratio of 9 to 6 is 3:2, that of the musical fifth. The ratio of 12 to 9 is 4:3, that of the fourth. The fifth being a larger span than the fourth, the latter is what Archytas speaks of as the lesser interval determined by the ratio of the greater numbers (12 and 9).

(ii) The three numbers 6, 12, 24 are in geometrical proportion, and 12 is the geometrical mean between 6 and 24, because the ratio of 24 to 12 is 2:1, the ratio of the octave. So, as Archytas puts it, the interval made by the greater terms (24 and 12) is equal to the interval made by the lesser (12 and 6) — an octave in both cases.

(iii) The three numbers 12, 8, 6 are in harmonic proportion, and 8 is the harmonic mean between 12 and 6, because 8 - 6 = 2 and 12 - 8 = 4: the difference in each case is a third part of the relevant extreme term, since 2 is a third part of 6 and 4 is a third part of 12 (in modern fractional notation, 2 = 6/3 and 4 = 12/3). Here the greater terms (12 and 8) make the greater interval, because the ratio of 12 to 8 is 3:2, the fifth, while the interval represented by 8:6 is 4:3, the fourth.

To explain how the three means were put to use in Greek music theory, I call on Andrew Barker (square bracketed additions mine):

The series 6, 12, 24 etc., in geometric proportion, represents a sequence of notes an octave apart. If we take the first two numbers and insert the arithmetic mean, we get 6, 9, 12, the octave being divided into a fifth [because 9:6 is 3:2] followed by a fourth [because 12:9 is 4:3]. A harmonic mean inserted between the original terms gives 6, 8, 12, dividing the octave into a fourth [because 8:6 is 4:3] followed by a fifth [because 12:8 is 3:2]. When the two sequences are combined, 6, 8, 9, 12, they yield two fourths [8:6 is 4:3 and 12:9 is 4:3] separated by the 'tone' of ratio 9:8, and can represent the fixed notes bounding a pair of disjoined tetrachords. [A tetrachord is a fourth, the upper and lower notes of which are fixed, but not the notes inserted in between. By varying the latter — in particular the distance of the highest from the upper bound — different musical 'genera' were produced: the enharmonic, the chromatic, the diatonic. Thus the tetrachord is a basic unit of scalar organisation.]\(^{69}\)

These are the fundamental relations on which all the complex structures of Pythagorean and Platonic harmonics are built.\(^{70}\)

Finally, an excerpt from the passage in Plato's Timaeus where the Divine Craftsman constructs the soul of the world as an elaborate scale or attunement of 27 notes, starting from two sequences in geometric proportion (1, 2, 4, 8 and 1, 3, 9, 27):

‘Next he filled out the double and triple intervals, once again cutting off parts from the mixture\(^{71}\) and placing them in the intervening gaps, so that in each interval there were two means, the one exceeding [one extreme] and exceeded [by the other extreme] by the same part of the extremes themselves, the other exceeding [one extreme] and exceeded [by the other] by an equal number.’ (35c–36a)\(^{72}\)

This is Archytas' language for the harmonic and arithmetic means, but redirected to elucidate the harmonious structure of a non-

\(^{68}\) Tr. Barker, GMW II, p. 42, from whose notes ad loc. I borrow the illustrations that follow.

\(^{69}\) For this and further details, see GMW II, pp. 11–13.

\(^{70}\) GMW II, pp. 42–3, n. 59.

\(^{71}\) The recipe for the mixture is given at 35ab: (i) take the indivisible Being that is always unchangingly the same and mix with the divisible being that comes to be in bodies, (ii) likewise, mix indivisible Sameness with its divisible counterpart, and (iii) indivisible Difference with divisible difference, then (iv) blend all three ingredients into a unity. For present purposes, all we need to understand of this is that the ‘stuff’ from which soul is made has some sort of intermediate status between Forms and sensibles; in this respect it is comparable to the objects of mathematics. But it is not soul, properly speaking, until the appropriate music-mathematical organisation has been imposed upon it.

\(^{72}\) Tr. Barker, GMW II, p. 59.
sensible entity, the soul. 73 The World Soul in the first instance, but the Divine Craftsman will later give the same structure to the less pure mixture from which he makes human souls (41d, 43d). Glaucos' exclamation, 'You are speaking of a task which is superhuman (δαμότιν πράγμα), may be pregnant with more meaning than he realises.

What I propose we should take from all this is the idea that the concords can be derived by operations with what Archytas called 'the three means in music'. Concord is explained by proportion. And these operations can be redirected to the analysis of structures which have little or nothing to do with sound. Soul provides the non-sensible subject-matter for a harmonics of the inaudible. 74

If this seems too general to explain why certain numbers are concordant, not others, let me add a further conjecture, suggesting that Plato may owe more to Archytas' language than appears from the Timaeus passage just quoted. 75 In the Harmonics of Ptolemy (second century AD) we find a discussion of 'the principles adopted by the Pythagoreans in their postulates about the concords', which offers strictly mathematical reasons for the thesis that multiple and epimoric ratios are a better (ἁμείθων) kind of ratio than epimoric. They are 'better' because of the simplicity of the comparison between the two terms of the ratio. In the case of epimoric ratios like 3:2, the excess [of the greater over the smaller term] is a simple part [integral factor, namely 1] of each of the terms. Multiples like 2:1 are even finer because the smaller term is itself a simple part of the greater. No such straightforward comparison of the terms is possible with an epimeric like 7:3. This result can then be used to explain why notes in the 'better' ratios sound better to the ear (Ptolemy, Harmonics I 5, 11.1–12.7 Düring). If Andrew Barker is right in maintaining that Archytas is the only Pythagorean we can identify as a plausible source for Ptolemy's report, then here is a

73 Archytas' language, but the scale itself is Philolaus' diatonic. Archytas' own scalar divisions are more complicated, because designed to account for actual musical practice: Barker, GMW II, pp. 46–52.
74 So Burkert, Lore and Science, pp. 372–3.

striking precedent for Plato to embrace the idea of a mathematics which makes direct use of evaluative concepts like 'better', and musical concepts like 'concordant', without first deriving them from auditory experience. From Plato's standpoint, Archytas' fault would be his developing such a mathematics merely in order to explain, from above as it were, the auditory experience we enjoy.

Indeed, Plato's own account of why concordant intervals sound good to the ear is a strictly physical explanation given in a much later section of the Timaeus (80ab). Following Archytas (frag. 1 Diels-Kranz), Timaeus states that pitch depends on the velocity with which air is driven to the ear by the source of the sound: the faster the transmission, the higher the pitch (Timaeus 67ac). When the slower and the faster of two motions have a certain 'similarity', they are heard as a single 'blend' of high and low: 'Hence they provide pleasure (ἡδονή) to people of poor understanding, and delight (εὐδοκιμία) to those of good understanding, because of the imitation of the divine attunement that comes into being in mortal movements' (80b). 76 Pleasure as such is merely the perception of restoration processes in the body (64c–65b). But what delights a listener familiar with the harmonics of the World Soul is that the agreeable stimulation of concordant sounds is a sensuous realisation of the non-sensible concords in the divine attunement. As the poet said, 'Heard melodies are sweet, but those unheard are sweeter'.

11. The ethical value of concord and attunement

This is the point at which to notice that concord has long been a value important to the overall argument of the Republic.

Way back in Book III, for example, Socrates laid down a rule that the material environment of the ideal city should be so designed that the young grow up surrounded by works of grace and beauty, whose impact on eye and ear will imperceptibly, from childhood on, guide them to likeness, to friendship, to concord (συμφωνία) with the beauty of reason (401cd). Their musical and

76 For the details, see Barker, GMW II, pp. 61–2, whose translation I have borrowed; Cornford goes badly wrong by applying 'because...' to both types of person.